A1

(i) Lorentz transformation

$x' = \gamma(x - vt) \qquad t' = \gamma(t - vx/c^2)$.

Since v and γ are constant,

$\delta x' = \gamma(\delta x - v\delta t) \qquad \delta t' = \gamma(\delta t - v\delta x/c^2)$

$\therefore \delta x'^2 - c^2\delta t'^2 = \gamma^2[\delta x^2 + v^2\delta t^2 - 2v\delta x\delta t - c^2\delta t^2 - \beta^2\delta x^2 + \cancel{2v\delta x\delta t}]$

$= \gamma^2(1-\beta^2)\delta x^2 - c^2\gamma^2(1-\beta^2)\delta t^2$

$= \delta x^2 - c^2\delta t^2 \qquad (\gamma^2(1-\beta^2) = 1)$

time-like interval: $c^2\delta t^2 > \delta x^2$ or $c|\delta t| > |\delta x|$.

When $\delta x = 0$; $\delta t = \delta\tau$ (the proper time) and $\delta s^2 = -c^2\delta\tau^2$

(ii) In general $\delta x'$ is zero in the equation $\delta x' = \gamma(\delta x - v\delta t)$ when $v = \delta x/\delta t$, or $v/c = \delta x/c\delta t$. ... (i)

Now since $c|\delta t| > |\delta x|$ for a time-like interval it is always possible to satisfy (i) with $v < c$.

Also, $\delta t' = \gamma(\delta t - v\delta x/c^2) = \gamma\delta t\left(1 - \frac{v\delta x}{c^2\delta t}\right)$.

With v given by (i)

$\delta t' = \gamma\delta t\left(1 - \underbrace{\frac{\delta x^2}{c^2\delta t^2}}_{\text{less than 1 by def}^n}\right)$.

$\therefore$ $\delta t'$ always has the same sign as δt.

Q2 (a) Consider a particle moving in an inertial frame S. Suppose the particle is instantaneously at rest in frame S'. Then $dx_\mu'^2 = dx_1'^2 + dx_2'^2 + dx_3'^2 - c^2 dt^2 = -c^2 d\tau^2$ because in S' $dx_i' = 0$ and $dt = d\tau$ (the proper time). Now c^2 and $dx_\mu'^2$ are invariants and therefore so is $d\tau$.

(b) let $B'_{\mu\nu} = \dfrac{\partial A'_\mu}{\partial x'_\nu} = \dfrac{\partial x_\lambda}{\partial x'_\nu}\dfrac{\partial A'_\mu}{\partial x_\lambda}$ (chain rule) ... (i)

Now $x_\lambda = a_{\phi\lambda} x'_\phi$ so $\dfrac{\partial x_\lambda}{\partial x'_\nu} = a_{\phi\lambda}\dfrac{\partial x'_\phi}{\partial x'_\nu} = a_{\phi\lambda}\delta_{\phi\nu}$
$= a_{\nu\lambda}$ ...(ii)

Also $A'_\mu = a_{\mu\phi} A_\phi$ so $\dfrac{\partial A'_\mu}{\partial x_\lambda} = a_{\mu\phi}\underbrace{\dfrac{\partial A_\phi}{\partial x_\lambda}}_{B_{\phi\lambda}} = a_{\mu\phi} B_{\phi\lambda}$ ...(iii)

So from (i)-(iii):

$B'_{\mu\nu} = a_{\nu\lambda} a_{\mu\phi} B_{\phi\lambda} = a_{\mu\phi} a_{\nu\lambda} B_{\phi\lambda}$ which proves that $B_{\phi\lambda}$ transforms like a second-rank 4-tensor.

(c)(i) $A_\mu = \dfrac{dU_\mu}{d\tau}$

(ii) $A_\mu = \dfrac{d}{d\tau}(\gamma_u \underline{u}, ic\gamma_u) = \dfrac{dt}{d\tau}\dfrac{d(\gamma_u \underline{u}, ic\gamma_u)}{dt}$

Q3 2.
(i) Let S and S' be frames in the standard configuration. Suppose a wave front passing O at time $t=0$ (i.e. passing O' at $t'=0$) is marked. When this wave front passes a point P in S counting starts (P is at rest in S). Similarly when it passes P' in S' counting starts (P' is at rest in S'). Counting stops at time t for P and at time t' for P' when $P(x,y,z)$ and $P'(x',y',z')$ are coincident. Then both P and P' count the same number of waves, N say. From the diagram

8 preamble

$\underline{k}\cdot\underline{r} = kr\cos\theta = kR$

5 algebra

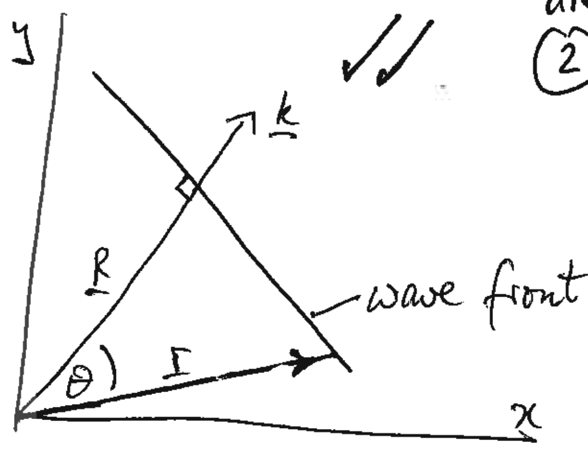
Therefore

$\underline{k}\cdot\underline{r} - \omega t = kR - \omega t$

$= -2\pi\nu(t - R/c)$

$= -2\pi\nu \times$ 'the counting time'

$= -2\pi N.$



diag. 2

Similarly, $\underline{k}'\cdot\underline{r}' - \omega' t' = -2\pi N.$

$\therefore \quad \underline{k}'\cdot\underline{r}' - \omega' t' = \underline{k}\cdot\underline{r} - \omega t$

Q3

(i) bookwork : PTO

(ii) Make an analogy with $x' = \gamma(x - vt) = \gamma(x - \beta ct)$

$y' = y$

$z' = z$

$ct' = \gamma(ct - \frac{v}{c}x) = \gamma(ct - \beta x)$

then $(\underline{k}, i\omega/c)$ transforms like $(\underline{r}, ict)$

so $k'_x = \gamma(k_x - \beta \frac{\omega}{c})$

$k'_y = k_y$

$k'_z = k_z$

$\omega'/c = \gamma(\omega/c - \beta k_x)$

(iii) Because k_μ is a 4-vector its "length"

$k_\mu^2 = k^2 - \omega^2/c^2$ is an invariant.

$\therefore$ $k_\mu^2 = k'^2_\mu$ or

$k^2 - \omega^2/c^2 = k'^2 - \omega'^2/c^2$

$$= \gamma_u \left(\frac{d\gamma_u}{dt} \underline{u} + \gamma_u \frac{d\underline{u}}{dt} \, , \, ic \frac{d\gamma_u}{dt} \right) \quad --- \; (*)$$

Now $\gamma_u = (1 - u^2/c^2)^{-1/2}$ so $\dfrac{d\gamma_u}{dt} = -\frac{1}{2}(1-u^2/c^2)^{-3/2}\left(-\dfrac{2\underline{u}\cdot\frac{d\underline{u}}{dt}}{c^2}\right)$

$$= \frac{\underline{u}\cdot\underline{a}/c^2}{(1-u^2/c^2)^{3/2}}$$

$$= \gamma_u^3 \, \underline{u}\cdot\underline{a}/c^2 \quad \ldots (**)$$

Substituting $(**)$ in $(*)$ gives:

$$A_\mu = \gamma_u \left(\gamma_u \underline{a} + \gamma_u^3 \frac{(\underline{u}\cdot\underline{a})\underline{u}}{c^2} \, , \, i\gamma_u^3 \frac{(\underline{u}\cdot\underline{a})}{c} \right)$$

$$= \gamma_u^2 \left(\underline{a} + \gamma_u^2 \frac{(\underline{u}\cdot\underline{a})\underline{u}}{c^2} \, , \, i\gamma_u^2 \frac{(\underline{u}\cdot\underline{a})}{c} \right)$$

or

$$\underline{A_\mu} = \underbrace{\gamma_u^2 \left(\underline{a} + \gamma_u^2 (\underline{\beta}\cdot\underline{a})\underline{\beta} \, , \, i\gamma_u^2 \underline{\beta}\cdot\underline{a} \right)} \qquad \underline{\beta} = \underline{u}/c .$$

Q4 EITHER

(a) Begin with $\underline{F}'_{\parallel} = \dfrac{(\underline{F}'\cdot\underline{v})\underline{v}}{v^2} = \dfrac{\gamma(\underline{F}_{\parallel}\cdot\underline{v} - v^2/c^2\,\underline{F}\cdot\underline{u})\underline{v} + \overbrace{\underline{F}_{\perp}\cdot\underline{v}}^{=0 \text{ by defn}}}{\gamma v^2(1-u_{\parallel}v/c^2)}$

So $\underline{F}'_{\parallel} = \dfrac{\cancel{\gamma}[F_{\parallel}v\underline{v} - v^2/c^2(\underline{F}_{\parallel}+\underline{F}_{\perp})\cdot(\underline{u}_{\parallel}+\underline{u}_{\perp})]\underline{v}}{\cancel{\gamma}v^2(1-u_{\parallel}v/c^2)}$ ... (*)

Now $F_{\parallel}\underline{v} = \underline{F}_{\parallel}v$ and (*) becomes:

$$\underline{F}'_{\parallel} = \frac{\underline{F}_{\parallel}v^2 - v^2/c^2(F_{\parallel}u_{\parallel} + \underline{F}_{\perp}\cdot\underline{u}_{\perp})\underline{v}}{v^2(1-u_{\parallel}v/c^2)}$$

$$= \frac{\underline{F}_{\parallel}v^2(1-u_{\parallel}v/c^2) - v^2/c^2(\underline{F}_{\perp}\cdot\underline{u}_{\perp})\underline{v}}{v^2(1-u_{\parallel}v/c^2)}$$

$$= \underline{F}_{\parallel} - \frac{(\underline{F}_{\perp}\cdot\underline{u}_{\perp})\underline{v}}{(c^2-u_{\parallel}v)}$$

b) worked example in notes: PTO.

Q4 OR

(a) $$\underline{F}' = \frac{\gamma(\underline{F}_{\parallel} - \underline{v}/c^2\, \underline{F}\cdot\underline{u}) + \underline{F}_{\perp}}{\gamma(1 - u_{\parallel} v/c^2)}$$

Now $\underline{F}\cdot\underline{u} = (\underline{F}_{\parallel} + \underline{F}_{\perp})\cdot(\underline{u}_{\parallel} + \underline{u}_{\perp}) = F_{\parallel} u_{\parallel} + \underline{F}_{\perp}\cdot\underline{u}_{\perp}$

since $\underline{F}_{\parallel}\cdot\underline{u}_{\perp} = \underline{F}_{\perp}\cdot\underline{u}_{\parallel} = 0$ by defn. So

$$\underline{F}' = \frac{\gamma\left[\underline{F}_{\parallel} - \underline{v}/c^2(F_{\parallel}u_{\parallel} + \underline{F}_{\perp}\cdot\underline{u}_{\perp})\right] + \underline{F}_{\perp}}{\gamma(1 - u_{\parallel}v/c^2)}$$

But $\underline{v} F_{\parallel} = v \underline{F}_{\parallel}$ and so

$$\underline{F}' = \frac{\gamma\, \underline{F}_{\parallel}(1 - u_{\parallel}v/c^2) - \overbrace{\gamma\underline{v}\, \underline{F}_{\perp}\cdot\underline{u}_{\perp}/c^2}^{(*)} + \underline{F}_{\perp}}{\gamma(1 - u_{\parallel}v/c^2)}$$

Now the term labelled with an (*) above is clearly part of $\underline{F}_{\parallel}$ since $\underline{F}_{\parallel}$ is parallel to $\underline{v}$ by definition. Then

$$\underline{F}'_{\parallel} = \underline{F}_{\parallel} - \frac{\underline{v}(\underline{F}_{\perp}\cdot\underline{u}_{\perp})}{(c^2 - u_{\parallel}v)} \quad \text{and} \quad \underline{F}'_{\perp} = \frac{\underline{F}_{\perp}}{\gamma(1 - u_{\parallel}v/c^2)}$$

(b) Consider the frame S' in which the charges are at rest. In this frame the charges repel each other with a force given by Coulomb's law.

$$F'_x = 0 \qquad F'_y = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{r^2} \qquad F'_z = 0$$

In a frame S moving with velocity $\underline{v} = v\hat{\underline{x}}$ relative to S', the force between the charges is given by the transformation

$$\underline{F}_{\parallel} = \underline{F}'_{\parallel} \quad \text{and} \quad \underline{F}_{\perp} = \underline{F}'_{\perp}/\gamma .$$

So

$$F_x = F'_x = 0$$

$$F_z = F'_z = 0$$

$$F_y = F'_y/\gamma = \frac{1}{4\pi\varepsilon_0}\frac{1}{\gamma}\frac{q^2}{r^2}$$

(assuming the invariance of q and $\underline{r}'_{\perp} = \underline{r}_{\perp}$)

$$= \frac{1}{4\pi\varepsilon_0}(1 - v^2/c^2)^{1/2}\frac{q^2}{r^2}$$

(b) • In the non-relativistic limit the force between the charges reduces to Coulomb's law.

• As $v \to c$, the repulsive electric force is almost cancelled by an attractive magnetic force. This proves that the force between these two charges will be repulsive in all inertial reference frames.