Surname & initials:

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Electromagnetism Test1 (2016) : Phys365

Time: 50 minutes Detailed Model Answer. **Total marks:** 50

General Instructions

- Check that you have 5 pages.
- Answer ALL questions.
- Candidates are reminded to be as thorough as possible and to write legibly.

Question 1

Write down the electric potential of a point electric dipole at the origin, and then use Cartesian tensors to prove that the field of this dipole is given by

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}}{r^3}. \quad \ldots (I)$$

(10)

$$V(\underline{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\underline{p}\cdot\underline{r}}{r^3} \quad \text{and} \quad \underline{E} = -\underline{\nabla} V$$

So $E_i = -\nabla_i V = -\frac{1}{4\pi\varepsilon_0}\nabla_i\left(\frac{p_j r_j}{r^3}\right) = -\frac{p_j}{4\pi\varepsilon_0}\nabla_i(r_j r^{-3})$

since $\underline{p}$ does not depend on position.

Applying the product rule gives:

$$E_i = -\frac{p_j}{4\pi\varepsilon_0}\left[(\nabla_i r_j) r^{-3} + r_j \nabla_i r^{-3}\right] = -\frac{p_j}{4\pi\varepsilon_0}\left[\delta_{ij} r^{-3} + r_j \frac{\partial}{\partial r_i} r^{-3}\right]$$

Now $\frac{\partial r^{-3}}{\partial r_i} = \frac{\partial r}{\partial r_i}\frac{\partial r^{-3}}{\partial r} = \frac{r_i}{r}(-3r^{-4}) = -3r_i r^{-5}$

So $E_i = -\frac{p_j}{4\pi\varepsilon_0}\left[\delta_{ij} r^{-3} - 3r_i r_j r^{-5}\right] \overset{\text{contracting}}{=} \frac{1}{4\pi\varepsilon_0}\left[3r_i r_j p_j r^{-5} - p_i r^{-3}\right]$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{3r_i(\underline{r}\cdot\underline{p}) - r^2 p_i}{r^5}\right].$$

Since this is true for $i = x$, y and z the vector form (I) follows immediately.

Question 2

Consider two coaxial conducting cylinders each of length L having radii a and b. Suppose $a < b$ and $L \gg b$. The space between the cylinders contains air. The inner cylinder is maintained at a positive potential V_0 with respect to the outer cylinder which is connected to ground.

(i) Use Laplace's equation to show that the electric potential $V(r)$ in the region $a \leq r \leq b$ is given by

$$V(r) = V_0 \frac{\ln(b/r)}{\ln(b/a)}.$$

(11)

(ii) Hence calculate the energy stored in the field between the cylinders. You may assume that the energy density is given by $\frac{1}{2}\epsilon_0 \mathbf{E} \cdot \mathbf{E}$ where all symbols have their usual meaning. (6)

(iii) Now suppose that the space between the conductors is filled with a homogeneous fluid having uniform conductivity σ. Derive an expression in terms of σ, L, a and b for the resistance R of the fluid. (5)

(i) Because of the symmetry of the problem V is a function of r only. Using the identity and Laplace's eq^n $\nabla^2 V = 0$ gives

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right) = 0;$$

which implies that $r\frac{dV}{dr}$ is a constant (α say). So

$$r\frac{dV}{dr} = \alpha \quad \text{or} \quad dV = \alpha\, dr/r.$$

Integrating gives $V = \alpha \ln r + \beta$ where β is a constant. We can use the boundary conditions to calculate α and β.

$V(r=a) = V_0 \quad : \quad V_0 = \alpha \ln a + \beta$

$V(r=b) = 0 \quad : \quad 0 = \alpha \ln b + \beta$ or $\beta = -\alpha \ln b$

So $V_0 = \alpha(\ln a - \ln b) = \alpha \ln a/b$

or $\alpha = V_0 / \ln(a/b)$ and $\beta = -\dfrac{V_0 \ln b}{\ln \frac{a}{b}}$

o $\quad V(r) = V_0 \dfrac{\ln r/b}{\ln a/b} = V_0 \dfrac{\ln b/r}{\ln b/a}$.

i) $\underline{E} = -\dfrac{\partial V}{\partial r}\hat{\underline{r}} = -\dfrac{\alpha}{r}\hat{\underline{r}} = -\dfrac{V_0/r}{\ln a/b}\hat{\underline{r}}$

$$= \frac{V_0/r}{\ln b/a}\hat{\underline{r}}$$

Integrate in cylindrical shells having volume $dv' = 2\pi r L\, dr$. Then

$$U = \frac{1}{2}\varepsilon_0 \int_a^b \left(\frac{V_0}{r \ln b/a}\right)^2 2\pi r L\, dr$$

io $\quad U/L = 2\pi \dfrac{\varepsilon_0}{2} \dfrac{V_0^2}{(\ln b/a)^2} \displaystyle\int_a^b \frac{dr}{r} = \dfrac{\pi \varepsilon_0 V_0^2}{\ln b/a}$

(iii) $I = \int_S \underline{J} \cdot d\underline{a}$ by definition

where $\underline{J} = \sigma \underline{E}$ (the microscopic form of ohm's law). So

$$I = \sigma \int_S \underline{E} \cdot d\underline{a} \quad \text{where } d\underline{a} = r\, d\phi\, dz\, \hat{\underline{r}}$$

So $$I = \frac{\sigma V_0}{\ln b/a} \int_S \frac{\cancel{r}\, d\phi\, dz}{\cancel{r}} \quad \text{since } \hat{\underline{r}} \cdot \hat{\underline{r}} = 1$$

$$= \frac{\sigma V_0}{\ln b/a} \int_0^L \int_0^{2\pi} d\phi\, dz$$

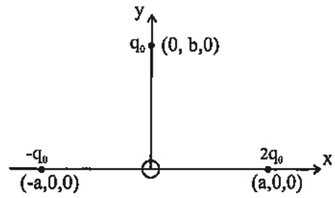
$$= \frac{2\pi \sigma L V_0}{\ln b/a}$$

Now $R = \frac{V_0}{I}$ and so $R = \frac{\ln(b/a)}{2\pi\sigma L}$

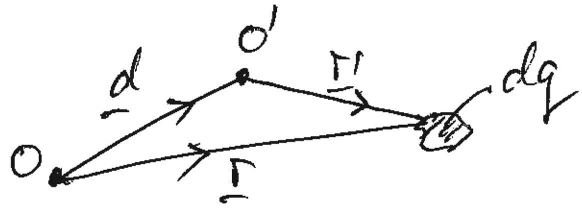
Question 3

(a) Prove that the electric dipole moment of an arbitrary charge distribution depends, in general, on the choice of origin. (6)

(b) Calculate the two lowest-order multipole moments for the distribution of point charges shown in the diagram alongside. Clearly show your working. (4)



(a) Consider a second origin O' displaced by an amount $\underline{d}$ from O as shown in the diagram. Let dq be an arbitrary element of charge whose position relative to O is $\underline{r}$ (and $\underline{r}'$ relative to O'). By defⁿ



$$\underline{P}_0 = \int \underline{r}\, dq \quad \text{where} \quad \underline{r} = \underline{d} - \underline{r}' \text{ (see diagram)}$$

$$\text{so } \underline{P}_0 = \int (\underline{d} - \underline{r}')\, dq' = \int \underline{d}\, dq' - \int \underline{r}' dq'$$

$$= \underline{d}\, Q - \underline{P}_{0'} .$$

since $\underline{d}$ is a constant displacement.

Here Q is the net charge of the distribution.

We see from this eqⁿ that $\underline{P}_0 = \underline{P}_{0'}$ only if $Q = 0$. Otherwise $\underline{P}$ is origin dependent.

(b) $Q = -q_0 + q_0 + 2q_0 = 2q_0$

$\underline{p} = \sum_{\alpha=1}^{n} \underline{r}'^{(\alpha)} q^{(\alpha)}$ for a discrete distribution of charge.

So $\underline{p} = (-q_0)(-a,0,0) + q_0(0,b,0) + 2q_0(a,0,0)$

$= (3q_0 a, q_0 b, 0)$

or $\underline{p} = 3q_0 a \, \hat{\underline{x}} + q_0 b \, \hat{\underline{y}}$

Question 4

(i) A neutral charge distribution having density $\rho(r)$ is placed in an external electrostatic field $\mathbf{E}(\mathbf{r})$. Derive an expression for the leading term in the multipole expansion for the force exerted on the distribution. (5)

(ii) Stating any assumptions which you make, use your answer to (i) to derive an expression for the potential energy U of a point electric dipole in an external electrostatic field $\mathbf{E}$. (3)

(i) $\underline{F} = \int \underline{E}\, dq = \int_v \underline{E}(\underline{r}')\rho(\underline{r}')\,dv'$

So $F_i = \int_v E_i\, \rho(\underline{r}')\,dv'$ where

$$E_i(\underline{r}') = (E_i)_0 + (\nabla_j E_i)_0 r_j' + \frac{1}{2!}(\nabla_k \nabla_j E_i)_0 r_k' r_j' + \cdots$$

$$\therefore\ F_i = \int_v \left[(E_i)_0 + (\nabla_j E_i)_0 r_j' + \cdots\right]\rho(\underline{r}')\,dv'$$

$$= E_i \underbrace{\int_v \rho(\underline{r}')\,dv'}_{=0 \text{ since the distribution is neutral}} + (\nabla_j E_i)\underbrace{\int_v r_j' \rho(\underline{r}')\,dv'}_{p_j \text{ by def}^n} + \cdots$$

$$= (\nabla_j E_i)\, p_j = (\underline{p}\cdot\underline{\nabla})E_i \quad \text{or} \quad \underline{F} = (\underline{p}\cdot\underline{\nabla})\underline{E}$$

(ii) Assumption: for a rigid distribution $\underline{F} = (\underline{p}\cdot\underline{\nabla})\underline{E} = \underline{\nabla}(\underline{p}\cdot\underline{E})$

Since electrostatic forces are conservative $\underline{F} = -\underline{\nabla}U$

Comparing these two results shows that

$$U = -\underline{p}\cdot\underline{E}$$