A1

(a) (i) $\nabla^2 V = 0$. Harmonic functions.

(ii) $\nabla^2 V = \nabla^2(\alpha_1 V_1 + \alpha_2 V_2) = \alpha_1 \nabla^2 V_1 + \alpha_2 \nabla^2 V_2 = 0$ since $\nabla^2 V_1 = \nabla^2 V_2 = 0$. So $\nabla^2 V = 0$ as required.

(b)(i) $\underline{E} = -\underline{\nabla} V$ or $E_i = -\nabla_i V = -\nabla_i (V_0 - E_{0j} r_j)$

$= E_{0j} \nabla_j r_i = E_{0j} \delta_{ij} = E_{0i}$. $\therefore$ $\underline{E} = \underline{E}_0$

(ii) $\underline{B} = \underline{\nabla} \times \underline{A}$ or $B_i = (\underline{\nabla} \times \underline{A})_i = \varepsilon_{ijk} \nabla_j A_k$ where

$A_k = \frac{\mu_0}{4\pi} \varepsilon_{klm} M_l r_m r^{-3}$. So

$$B_i = \frac{\mu_0}{4\pi} M_l \varepsilon_{ijk} \varepsilon_{klm} \nabla_j (r_m r^{-3}) \qquad (M_l \text{ independent of position})$$

$$= \frac{\mu_0}{4\pi} M_l \varepsilon_{kij} \varepsilon_{klm} [r^{-3} \nabla_j r_m + r_m \nabla_j r^{-3}] \qquad \left(\begin{array}{l}\text{cyclic props of } \varepsilon \text{ tensor} \\ \text{product rule}\end{array}\right)$$

$$= \frac{\mu_0 M_l}{4\pi} (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) [r^{-3}\delta_{jm} - 3 r_j r_m r^{-5}] \qquad \dots (*)$$

because $\nabla_j r^{-3} = \frac{\partial r^{-3}}{\partial r_j} = \frac{\partial r}{\partial r_j}\frac{\partial r^{-3}}{\partial r} = -3r^{-4}\frac{r_j}{r}$

Expanding $(*)$ and contracting tensor subscripts give:

$$B_i = \frac{\mu_0}{4\pi} M_l (\cancel{3r^{-3}\delta_{il}} - \cancel{3r^{-3}\delta_{il}} - r^{-3}\delta_{il} + 3 r_i r_l r^{-5})$$

$$= \frac{\mu_0}{4\pi} \frac{[3(\overbrace{M_l r_l}^{\underline{M}\cdot\underline{r}}) r_i - M_i r^2]}{r^5} \qquad \dots (**)$$

Now $(**)$ is true for $i = x, y$ and z. So

$$\underline{B} = \frac{\mu_0}{4\pi} \frac{3(\underline{M}\cdot\underline{r})\underline{r} - r^2 \underline{M}}{r^5} \quad \text{or} \quad \frac{\mu_0}{4\pi} \frac{3(\underline{M}\cdot\hat{\underline{r}})\hat{\underline{r}} - \underline{M}}{r^3}$$

A2

(a) $\oint_S \underline{B} \cdot d\underline{a} = 0$

Applying the divergence th^m gives $\int_v \underline{\nabla} \cdot \underline{B} \, dv = 0$ (*)

Since (*) is true for arbitrary v we have $\underline{\nabla} \cdot \underline{B} = 0$.

(b)(i) Take $\underline{\nabla} \cdot$ of both sides:

$$\underline{\nabla} \cdot (\underline{\nabla} \times \underline{B}) = \mu_0 \underline{\nabla} \cdot \underline{J} = 0 \qquad (\text{div (curl) is zero})$$

which is not true since $\underline{\nabla} \cdot \underline{J} = -\partial \rho / \partial t$ in general.

(ii) law of conservation of electric charge.

(iii) We need to construct a new vector which has the following properties:

→ reduces to $\underline{J}$ in the steady state,

→ does not violate charge conservation, ... (*)

→ reduces to $\underline{J}$ in the steady state.

$$\underline{\nabla} \cdot \underline{J} + \frac{\partial \rho}{\partial t} = 0 \quad \text{with } \rho = \varepsilon_0 \underline{\nabla} \cdot \underline{E} \quad (\text{Gauss' law})$$

$$\underline{\nabla} \cdot \underline{J} + \varepsilon_0 \frac{\partial}{\partial t} (\underline{\nabla} \cdot \underline{E}) = \underline{\nabla} \cdot \underline{J} + \underline{\nabla} \cdot \varepsilon_0 \frac{\partial \underline{E}}{\partial t} = \underline{\nabla} \cdot \underbrace{\left(\underline{J} + \varepsilon_0 \frac{\partial \underline{E}}{\partial t} \right)}_{\underline{J}_{new}} = 0.$$

$\underline{J}_{new}$ satisfies (*) and is the vector we seek.

Replacing $\underline{J}$ in Ampère's magnetostatic eq^n gives

$$\underline{\nabla} \times \underline{B} = \mu_0 \left(\underline{J} + \varepsilon_0 \frac{\partial \underline{E}}{\partial t} \right) \quad \text{as required.}$$

A3

$$\begin{rcases} \nabla\cdot\underline{E} = 0 \\ \nabla\times\underline{E} = -\partial\underline{B}/\partial t \\ \nabla\cdot\underline{B} = 0 \\ \nabla\times\underline{B} = \varepsilon_0\mu_0\dfrac{\partial\underline{E}}{\partial t} \end{rcases}$$ Maxwell's equations in a source-free vacuum

Suppose
$$\begin{rcases} \underline{E} = E_0 e^{i(\underline{k}\cdot\underline{r}-\omega t)}\,\hat{\underline{\varepsilon}}_1 \\ \underline{B} = B_0 e^{i(\underline{k}\cdot\underline{r}-\omega t)}\,\hat{\underline{\varepsilon}}_2 \end{rcases}$$ Here E_0 and B_0 are constants, possibly complex and $\hat{\underline{\varepsilon}}_1$, $\hat{\underline{\varepsilon}}_2$ are real constant unit vectors.

$$\nabla\cdot\underline{E} = E_0 e^{-i\omega t}\,\nabla\cdot\left(e^{i\underline{k}\cdot\underline{r}}\,\hat{\underline{\varepsilon}}_1\right) = 0$$

$$= E_0 e^{-i\omega t}\left[\underbrace{\hat{\underline{\varepsilon}}_1\cdot\nabla e^{i\underline{k}\cdot\underline{r}}}_{\hat{\underline{\varepsilon}}_1\cdot i\underline{k}e^{i\underline{k}\cdot\underline{r}}} + e^{i\underline{k}\cdot\underline{r}}\underbrace{\nabla\cdot\hat{\underline{\varepsilon}}_1}_{=0 \text{ since } \underline{\varepsilon}_1 \text{ is constant}}\right] = 0 \quad \begin{pmatrix}\text{product}\\ \text{rule}\end{pmatrix}$$

so $\hat{\underline{E}}_1\cdot\underline{k}e^{i(\underline{k}\cdot\underline{r})}E_0 e^{i(\underline{k}\cdot\underline{r}-\omega t)} = 0$

or $\underline{E}\cdot\underline{k} = 0$ which proves $\underline{E}$ is $\perp^r$ to $\underline{k}$

Similarly

$\nabla\cdot\underline{B} = 0$ gives $\underline{B}\cdot\underline{k} = 0$ which proves that $\underline{B}$ is $\perp^r$ to $\underline{k}$

Since $\underline{E}$ and $\underline{B}$ are both transve to the direction of propagation the wave is transverse.

A4 (i) From Faraday - Maxwell:

$$\underline{\nabla} \times \underline{E} = -\partial \underline{B}/\partial t$$

where $\underline{\nabla} \times \underline{E} = \hat{\underline{x}}\left(\frac{\partial E_z}{\partial y} - \cancel{\frac{\partial E_y}{\partial z}}\right) + \hat{\underline{y}}\left(\cancel{\frac{\partial E_x}{\partial z}} - \cancel{\frac{\partial E_z}{\partial x}}\right) + \hat{\underline{z}}\left(\cancel{\frac{\partial E_y}{\partial x}} - \cancel{\frac{\partial E_x}{\partial y}}\right)$

0 since E_z does not depend on x.

$$= \hat{\underline{x}} \frac{\partial E_z}{\partial y}$$

Now $\frac{\partial E_z}{\partial y} = -kE_0 \sin(ky + \omega t)$

So $\frac{\partial B_x}{\partial t} = kE_0 \sin(ky + \omega t)$; $\frac{\partial B_y}{\partial t} = \frac{\partial B_z}{\partial t} = 0$

$$B_x = kE_0 \int \sin(ky + \omega t)\, dt$$

$$= -\frac{kE_0}{\omega} \cos(ky + \omega t) + \text{constant of integration } C$$

time - independent

Clearly C is not the field of a wave and we set it equal to 0. So

$$B_x = -\frac{E_0}{c} \cos(ky + \omega t) \qquad (c = \omega/k)$$

$$\underline{B} = \left(-\frac{E_0}{c} \cos(ky + \omega t), 0, 0\right)$$

(ii) $\langle \underline{S} \rangle = -\frac{1}{2} \sqrt{\frac{\mu_0}{\varepsilon_0}}\, |E_0|^2\, \hat{\underline{y}}$

A5

(a) $\underline{E}' \cdot \underline{B}' = E_x' B_x' + E_y' B_y' + E_z' B_z'$

$$= E_x B_x + \gamma^2 (E_y - vB_z)(B_y + \frac{v}{c^2} E_z) + \gamma^2 (E_z + vB_y)(B_z - \frac{v}{c^2} E_y)$$

$$= E_x B_x + \gamma^2 \left[E_y B_y + E_z B_z - \frac{v^2}{c^2} E_y B_y - \frac{v^2}{c^2} E_z B_z \right]$$

Since the remaining 4 terms cancel exactly.

So $\underline{E}' \cdot \underline{B}' = E_x B_x + E_y B_y \gamma^2 (1 - v^2/c^2) + E_z B_z \gamma^2 (1 - v^2/c^2)$

But $\gamma^2 (1 - v^2/c^2) = 1$ by definition, and

so $\underline{E}' \cdot \underline{B}' = E_x B_x + E_y B_y + E_z B_z = \underline{E} \cdot \underline{B}$ as reqd.

b) $E_x = 0$ $\quad B_x = 0$ $\quad \gamma = \frac{1}{\sqrt{1 - 1/4}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

$E_y = 0$ $\quad B_y = 0$

$E_z = 6 \times 10^6 \ Vm^{-1}$ $\quad B_z = 0$

$E_x' = 0$ $\quad B_x' = 0$

$E_y' = 0$ $\quad B_y' = \frac{2\sqrt{3}}{3} \left(0 + \frac{1}{2} \frac{1}{3 \times 10^8} 6 \times 10^6 \right) = \frac{2\sqrt{3} \times 10^{-2}}{3} \ T$

$E_z' = \frac{2\sqrt{3}}{3} (6 \times 10^6 + 0)$ $\quad B_z' = 0$

$= 4\sqrt{3} \times 10^6 \ Vm^{-1}$

So $\underline{E}' = 4\sqrt{3} \times 10^6 \ \hat{\underline{z}} \ Vm^{-1}$ and $\underline{B}' = \frac{2\sqrt{3}}{3} \times 10^{-2} \ \hat{\underline{y}} \ T$